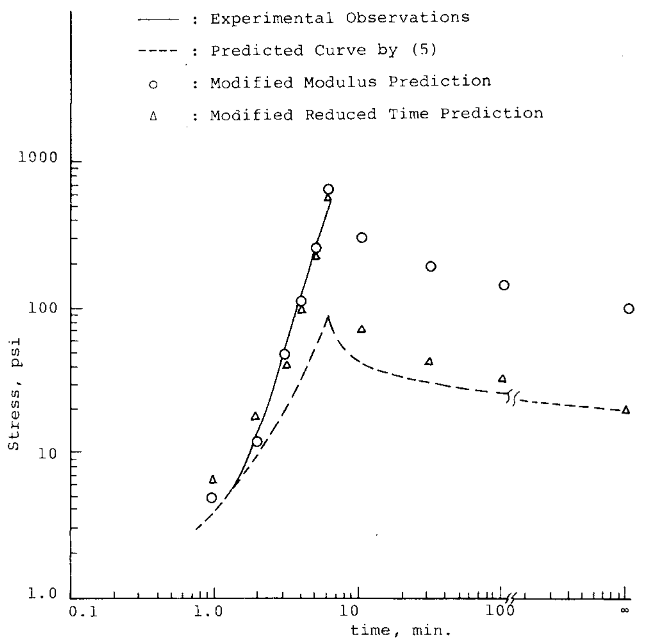

Fig. 1 Comparison of observation and prediction.

where a and c are constants to be determined. Apply Eqs. (3), (11), and (14) with Eq. (10); the stress response in the transient portion is

$$\sigma(t) = (R-\alpha k)\left[E_0 t - \frac{2E_1}{Kk}\sqrt{|Kk|e^{(b-KT_0)}}\, e^{-Kkt/2}\tan^{-1}\sqrt{|e^{-Kkt}-1|}\right] + \Delta\sigma(t) \tag{15}$$

where

$$\Delta\sigma(t) = af_1(t) + cf_2(t) \tag{16}$$

$$f_1(t) = -E_0 kt + \frac{2E_1}{K}\sqrt{|Kk|e^{(b-KT_0)}}\, e^{-Kkt/2} \times \tan^{-1}\sqrt{|e^{-Kkt}-1|} \tag{17}$$

$$f_2(t) = -E_0T_0kt + \frac{E_0k^2}{2}t^2 + \frac{2E_1}{K}\sqrt{|Kk|e^{(b-KT_0)}}\, e^{-Kkt/2}\left[T_0\tan^{-1}\sqrt{|e^{-Kkt}-1|} + k\int_0^t \tan^{-1}\sqrt{|e^{-Kk(t-\tau)}-1|}\,d\tau\right] \tag{18}$$

substitute the transient experimental data[3] into Eq. (15), through numerical calculations; the constants a and c for these materials are determined by using a least-square technique.[3] Then the transient and relaxation predictions by this approach are shown by circles in Fig. 1.

IV. Conclusions

When the predicted results are compared with observations, the following conclusions can be drawn.

1) The transient prediction by Eq. (5) will lead to a large error due to the coupled thermomechanical effects.

2) The modified reduced time approach predicts a consistent result with observations in the transient portion. But the prediction in the relaxation portion is questionable, since it predicts the same equilibrium value as predicted by Eq. (5).

3) The modified modulus approach will predict consistent results in both the transient and relaxation portions. Therefore, it may be considered to be the proper form to represent the coupled thermomechanical effects for highly filled solid propellants.

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Optimum Design of Laminated Plates under Axial Compression

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Introduction

RECENTLY, filamentary composite materials have been suggested for the primary structure of aircraft and spacecraft. The reason is primarily the weight savings that can be attained. Many design criteria must be considered in the design of composite materials for structures; the buckling criterion is one. Several theoretical and experimental papers have been published on the buckling of laminated composite plates under axial compression. Housner and Stein[1] calculated the optimum fiber direction of graphite-epoxy sandwich panels under axial compression, assuming that the angle of all the plies is the same.

This paper presents a method to design laminated plates with orthotropic layers under uniaxial and biaxial compression. The design criterion is the buckling stress. Each layer of the plate is assumed to have the same thickness and an equal number of fibers of the same kind in the $+\alpha_i$ and $-\alpha_i$ directions with respect to the x coordinate in the same type of matrix. Therefore, each layer can be considered to be orthotropic. Inhomogeneity in the direction of the thickness of the plate (stacking sequence) is taken into account in the calculation; therefore, the present calculation allows almost complete freedom for the choice of all the ply angles.

The present problem is to find the fiber directions of all the layers that give the highest buckling stress; therefore, one has to solve an unconstrained maximization problem. The objective function and the design variables are the critical buckling stress and the fiber directions, respectively. Preassigned parameters are the material properties, the thickness of each layer, the number of layers, the aspect ratio of the plates, and the ratio of the force per unit width in the y

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direction to the force per unit width in the x direction. The optimization technique used is Powell's method (conjugate direction technique). This method is one of the best ones to find the optimum without using the derivatives of the objective function.

Derivation of Buckling Formula

Extensional, coupling, and bending stiffnesses,[2] expressed as A_{ij}, B_{ij}, and D_{ij}, respectively, are calculated for the present case as:

$$A_{ij}=h\{(\bar{Q}_{ij})_1+(\bar{Q}_{ij})_2+\ldots+(\bar{Q}_{ij})_{N-1}+(\bar{Q}_{ij})_N\} \tag{1a}$$

$$2B_{ij}=h^2[(\bar{Q}_{ij})_1\{-N+1\}+(\bar{Q}_{ij})_2\{-N+3\} + (\bar{Q}_{ij})_3\{-N+5\}+\ldots+(\bar{Q}_{ij})_{N-1}\{-N+(2N-3)\} + (\bar{Q}_{ij})_N\{-N+(2N-1)\}] \tag{1b}$$

$$3D_{ij}=h^3\Big\{(\bar{Q}_{ij})_1\Big[\Big\{-\frac{N}{2}+1\Big\}^3-\Big\{-\frac{N}{2}\Big\}^3\Big] + (\bar{Q}_{ij})_2\Big[\Big\{-\frac{N}{2}+2\Big\}^3-\Big\{-\frac{N}{2}+1\Big\}^3\Big]+\ldots + (\bar{Q}_{ij})_{N-1}\Big[\Big\{-\frac{N}{2}+(N-1)\Big\}^3-\Big\{-\frac{N}{2}+(N-2)\Big\}^3\Big] + (\bar{Q}_{ij})_N\Big[\Big\{-\frac{N}{2}+N\Big\}^3-\Big\{-\frac{N}{2}+(N-1)\Big\}^3\Big]\Big\} \tag{1c}$$

where h is the thickness of each layer, N the total number of layers, $\bar{Q}_{ij}$ the transformed reduced stiffness, and the subscript of $(\bar{Q}_{ij})$ the number of each layer. It should be noted that $\bar{Q}_{16}$ and $\bar{Q}_{26}=0$ for the present problem.

Whitney and Leissa[3] derived equilibrium equations for the general laminated plates. These equations are now simplified for the present problem as follows:

$$A_{11}u_{,xx}+A_{66}u_{,yy}+(A_{12}+A_{66})v_{,xy}-B_{11}w_{,xxx} - (B_{12}+2B_{66})w_{,xyy}=0 \tag{2a}$$

$$(A_{12}+A_{66})u_{,xy}+A_{66}v_{,xx}+A_{22}v_{,yy}-(B_{12}+2B_{66})w_{,xxy} - B_{22}w_{,yyy}=0 \tag{2b}$$

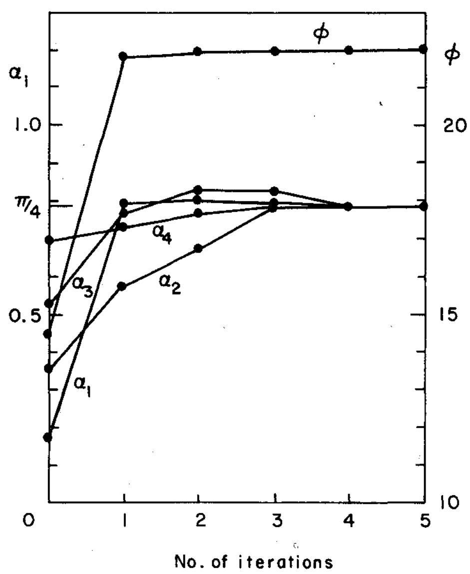

Fig. 1 Variation of α_i and ϕ with number of iterations.

$$D_{11}w_{,xxxx}+2(D_{12}+2D_{66})w_{,xxyy}+D_{22}w_{,yyyy}-B_{11}u_{,xxx} - (B_{12}+2B_{66})u_{,xyy}-(B_{12}+2B_{66})v_{,xxy}-B_{22}v_{,yyy} + \bar{N}_x w_{,xx}+\bar{N}_y w_{,yy}=0 \tag{2c}$$

where u, v, and w are displacements in the x, y, and z directions, respectively, and $\bar{N}_x$ and $\bar{N}_y$ are applied compressive force per unit width. The buckling deformations are assumed as:

$$u=\bar{u}\cos(m\pi x/a)\sin(n\pi y/b) \tag{3a}$$

$$v=\bar{v}\sin(m\pi x/a)\cos(n\pi y/b) \tag{3b}$$

$$w=\bar{w}\sin(m\pi x/a)\sin(n\pi y/b) \tag{3c}$$

These deformations satisfy the simply supported boundary conditions at $x=0$, a and $y=0,b$ (S2 of Ref. 4). Substitution from Eqs. (3) into Eqs. (2) and the definition $\bar{N}_y=k\bar{N}_x$ gives the following buckling formula:

$$\frac{12\bar{N}_x b^2}{\pi^2 t^3 Q_{22}}=\frac{12(b/a)^2}{\pi^4 t^3 Q_{22}\{m^2+kn^2(a/b)^2\}} \times\left[T'_{33}+\frac{2T'_{12}T'_{23}T'_{13}-T'_{22}T'^{2}_{13}-T'_{11}T'^{2}_{23}}{T'_{11}T'_{22}-T'^{2}_{12}}\right] \tag{4}$$

where

$$T'_{11}=A_{11}m^2\pi^2+A_{66}n^2\pi^2(a/b)^2 \tag{5a}$$

$$T'_{12}=(A_{12}+A_{66})mn\pi^2(a/b) \tag{5b}$$

$$T'_{13}=B_{11}m^3\pi^3+(B_{12}+2B_{66})mn^2\pi^3(a/b)^2 \tag{5c}$$

$$T'_{22}=A_{66}m^2\pi^2+A_{22}n^2\pi^2(a/b)^2 \tag{5d}$$

$$T'_{23}=(B_{12}+2B_{66})m^2n\pi^3(a/b)+B_{22}n^3\pi^3(a/b)^3 \tag{5e}$$

$$T'_{33}=D_{11}m^4\pi^4+2(D_{12}+2D_{66})m^2n^2\pi^4(a/b)^2 + D_{22}n^4\pi^4(a/b)^4 \tag{5f}$$

To get the critical buckling stress, the smallest value of Eq. (4) must be found by a searching procedure involving integral values of m and n. The critical buckling stress is denoted by $(\bar{N}_x)_{cr}/t$, and a new symbol ϕ is introduced as

$$\phi=12(\bar{N}_x)_{cr}b^2/(\pi^2 t^3 Q_{22}) \tag{6}$$

For isotropic plates, ϕ is equal to 4 when $a/b=1$ and $k=0$.

Method of Optimization

The problem is to find the fiber directions that give the maximum critical buckling stress without any constraints; therefore, we can apply one of the unconstrained optimization techniques. Since the objective function, Eq. (4), is a rather complicated function of the design variables (fiber directions), optimization methods that do not use the derivatives are suitable for solving the problem. Powell's method[5] (conjugate direction technique) is selected for use since it is one of the best methods to find the optimum without using the derivatives. Starting values of fiber directions $(\alpha_1,\alpha_2,\ldots,\alpha_N)$ are necessary to begin the calculation, and new fiber directions that give the higher buckling stress are obtained after each iteration. In this process, the buckling formula must be minimized with respect to the integral numbers of half-waves in the x and y directions for the assigned fiber directions. Powell's method requires that the objective function be unimodal, but it is not known if the function is unimodal or

Table 1 Optimum fiber directions

No. of layers	k	a/b	Fiber directions, deg α_1	α_2	α_3	α_4	α_5	α_6	Reduced critical stress, ϕ
3	0	1.0	45	45	45				22.000
4	0	1.0	45	45	45	45			22.000
6	0	0.5	0	0	0	0	0	0	42.171
6	0	0.8	38	38	38	38	38	38	23.154
6	0	1.0	45	45	45	45	45	45	22.000
6*	0	1.25	49.9	51.0	48.6	48.8	51.0	49.9	23.116
6	0	2.0	45	45	45	45	45	45	22.000
6*	0.5	0.5	7.9	3.8	22.1	0.5	7.4	12.3	37.024
6	0.5	1.0	45	45	45	45	45	45	14.667
6*	0.5	2.0	67.1	56.4	56.2	55.5	64.0	61.4	12.556
6	1.0	1.0	45	45	45	45	45	45	11.000
6*	1.0	2.0	71.6	68.1	77.5	61.2	71.1	74.1	8.051

not. Therefore, trials with several starting points are desirable.

Numerical Examples

Numerical calculations were made for laminated plates with three, four, and six layers. The plates considered have various aspect ratios and are under uniaxial or biaxial compression. Seven or eight different combinations of fiber directions are used to start the calculation. The computer code[6] developed by Powell was rearranged into the code written for the present problem.

The convergence limits for all the design variables were set equal to 0.1 deg, and the maximum step size multiplier[6] in single variable search was set equal to 10.0. The material consider is boron/epoxy.

$$E_1 = 2.11 \times 10^4 \text{ kg/mm}^2 \ (30 \times 10^6 \text{ psi})$$

$$E_2 = 2.11 \times 10^3 \text{ kg/mm}^2 \ (3 \times 10^6 \text{ psi})$$

$$\nu_{12} = 0.3,\ G_{12} = 7.03 \times 10^2 \text{ kg/mm}^2 \ (1 \times 10^6 \text{ psi})$$

The thickness of each layer is assumed to be 0.254 mm (0.01 in.). The buckling formula is a function of the half-waves in the x and y directions. Therefore, the number of half-waves in the x and y directions was varied from 1 to 10 and from 1 to 5, respectively, to get the buckling stress for the assigned fiber directions.

The numerical results for three- and four-layered plates with $a/b = 1$ and $k = 0$ showed that the results obtained do not depend on the starting values of fiber directions. To show the numerical convergence, an example is presented in Fig. 1. In this figure, the abscissa is the number of the iteration. Each iteration includes many function evaluations. The method was next applied to six-layered plates with various aspect ratios and load ratios k. A summary of all the numerical results is given in Table 1. In this table, asterisks show that the final results obtained depend on the starting values, and the values shown correspond to the highest critical buckling stress obtained among eight cases with different starting values. In these cases, the critical buckling stresses obtained are not much different from each other, but the fiber directions depend highly on the starting values. The fiber directions in the rows without an asterisk have no decimal, because almost all of the directions obtained for seven or eight cases are close to the values shown.

The computer used was IBM 360/67 and average cpu time to calculate eight cases for a six-layered plate under $k = 0$ was 158 s.

Acknowledgment

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Flexural Vibration of Orthotropic Cylindrical Shells in a Fluid Medium

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Introduction

The problem of free vibration of a cylindrical shell has interested many investigators since the time of Rayleigh. In recent years this interest has continued in the area of composite shells, due to their increasing importance in aerospace and hydrospace applications.

The first experimental determination of natural frequencies of cylindrical shells was made by Arnold and Warburton[1]

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